Mar 7.

last week : local invertibility

$$\mathcal{E}(f) = \left(\frac{\partial f}{\partial t} - E(f),\ f|_{t=0}\right)$$

$$D\mathcal{E}(f)\tilde{f} = \left(\frac{\partial \tilde{f}}{\partial t} - DE(f)(\tilde{f}),\ \tilde{f}|_{t=0}\right) \qquad (\star)$$

$$P(f) = DE(f) + L^*(f)\,L(f)$$

We saw that $\frac{\partial \tilde{f}}{\partial t} = P(f)\tilde{f}$ is parabolic

We can rewrite $(\star)$ as system of equations

$$\begin{cases} \frac{\partial \tilde{f}}{\partial t} - P(f)\tilde{f} + L^*(f)\,\tilde{g} = \tilde{h} \\ \frac{\partial \tilde{g}}{\partial t} - M(f)\tilde{f} = \tilde{k} \end{cases} \qquad (\star\star)$$

where $M(f) = DL(f)\{\tilde{f}, \frac{\partial f}{\partial t}\} - DL(f)\{E(f), \tilde{f}\} + DQ\,\tilde{f}$

with IC $\begin{cases} \tilde{f}|_{t=0} = \tilde{f}_0 \\ \tilde{g}|_{t=0} = L(f_0)\,\tilde{f}_0 \end{cases}$ and $\tilde{g} = L(f)\tilde{f}$.

It turns out $(\star)$ and $(\star\star)$ have the same solution.

$(\star)$ and $(\star\star)$ are equivalent : • $[(\star\star) \Leftarrow (\star)]$

Let $\tilde{\ell} = \tilde{g} - L(f)\tilde{f}$ then $\frac{\partial \tilde{\ell}}{\partial t} = L(f)\,L^*(f)\,\tilde{f}$.

and $\frac{d}{dt}\int_X |\tilde{\ell}|^2\, d\mu + 2\int |L^*(f)|^2\, d\mu = 0$

IC $\Rightarrow$ at time 0 $\tilde{\ell} = \tilde{g}_0 - L(f_0) f_0 = 0$

energy condition $\Rightarrow$ $\frac{d}{dt} \int |\tilde{\ell}|^2 \, d\mu < 0$ decreasing

$\Rightarrow$ $\tilde{\ell} \equiv 0 \quad \forall t.$

- [(☆) $\Rightarrow$ (☆☆)] easier.

Consider
$$\begin{cases} \frac{\partial f}{\partial t} = Pf + Lg + h \\ \frac{\partial g}{\partial t} = Mf + Ng + k \end{cases} \qquad (☆☆☆)$$

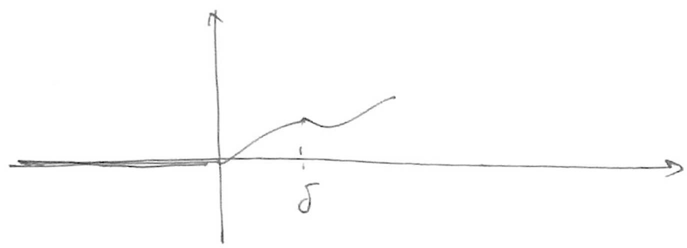
where P is parabolic, deg 2
L, M deg 1. N deg 0.

<u>Thm</u> Suppose $\frac{\partial f}{\partial t} - Pf = 0$ is parabolic. Then
for all given IC (f_0, g_0, k, h)
$\exists!$ (f, g) solves (☆☆☆) with IC $f|_{t=0} = f_0$
$g|_{t=0} = g_0$

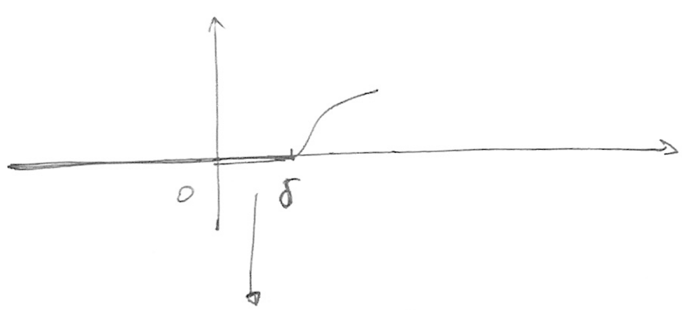
for Ricci flow. f is the metric
$g = L(f) F$ is the Bianchi term.

Using similar idea (Taylor series), we can assume
f, g, h, k vanish for $t \le 0$.

$$\begin{cases} \dfrac{\partial f}{\partial t} = Pf + Lg + h \\[2ex] \dfrac{\partial g_\delta}{\partial t}(t+\delta) = (Mf + Ng_\delta + h)(t) \end{cases} \qquad \binom{\star\star}{\star\star}$$



f



g

$g=0 \quad \dfrac{\partial f}{\partial t} = Pf + h.$ solve f using parabolicity

$\exists\, T_\delta$ s.t. (f, g_δ) solves $\binom{\star\star}{\star\star}$ for $t \in [0, T_\delta) \subseteq [0,1)$.

Claim
($|f|$, $|g_\delta|$ are bounded, the bounds are independent of δ) Then can take $(f, g_{\delta_k}) \xrightarrow{C^\infty} (f, g)$ as $k \to \infty$.

$$T_{\delta_k} \longrightarrow T > 0$$

To show the claim:

$$|f_t|_n = \int_X |f_t|^2 \, d\mu + \sum_{|\alpha| \le n} \int_X |D^\alpha f_t|^2 \, d\mu.$$

$$\|f\|_n^2 := \sum_{2j \le n} \left\| \left(\frac{\partial}{\partial t}\right)^j f_t \right\|_{n-2j}$$

time variable has different weight than spacial variables norm estimate arguement . . .

$\|f\|_n + \|g_\delta\|_n \leq C(\|P\|, \|L\|, \|M\|, \|N\|)$ indpt of δ.